J. Fluid Mech. (1967), *vol.* 28, *part* 3, *pp.* 463–479
Printed in Great Britain

The spin up of a stratified fluid

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(Received 19 May 1966)

The process by which a stratified, viscous fluid adjusts to small changes in the rotation rate of its container is studied. This paper treats the cases of homogeneous layers of different densities, as well as fluids which are continuously stratified.

It is shown that in several important cases the spin-up process, especially in the continuously stratified case, has a time scale which is very much longer than for homogeneous fluids, and that diffusion is the governing mechanism in the adjustment process.

In all cases the detailed problem, including a discussion of the side-wall boundary layers, is presented. Some novel features of the side-wall layers are discussed for the continuously stratified fluids, while in one case it is shown that *no* boundary layers appear during the transient approach to equilibrium.

1. Introduction

When a cylindrical vessel filled with a homogeneous, viscous, incompressible fluid rotating at a constant angular velocity Ω has its rotation rate slightly altered, the ensuing transient process by which the fluid adjusts to the new angular velocity of the container has been described by Greenspan & Howard (1963). They showed that the time taken for the fluid to adjust to the new state is $(L^2/\nu\Omega)^{\frac{1}{2}}$, where L is a characteristic dimension parallel to the axis of rotation and ν the kinematic viscosity of the fluid. The adjustment process is controlled by the suction of the fluid into Ekman boundary layers on the horizontal surfaces of the container, and the resulting secondary motion 'spins up' the fluid by vortex tube stretching parallel to the rotation axis.

A question of great interest is the effect that stratification has on this process, considering that the stratification could inhibit the secondary flow and alter the spin-up dynamics. This problem is clearly central to an understanding of the dissipative mechanism for atmospheric and oceanic motions of a large scale, where rotational effects are dominant.

This problem was studied by Holton (1965) and the reader is referred to his papers for an interesting discussion of the geophysical significance of this problem. Holton found that stratification could play an essential role in the 'spin-up' problem and he considered several problems in detail. He did not, however, consider in detail the role of the vertical containing boundaries of the fluid in his discussion of the spin up of contained fluids in the laboratory.

It is the purpose of this paper to reconsider this problem and show that a consideration of the side-wall boundary is an essential ingredient in a complete theory of the spin up of a *contained* stratified fluid. The analysis will centre on the dynamics of two layers of immiscible fluids and, further, it will be shown that there is an essential difference between the layer model and the problem of the spin up of a continuously stratified fluid. Although the analysis given here for *contained* fluids differs in certain important aspects from Holton's, his conclusions for the geophysically interesting case of the dissipation of motions in an unbounded fluid are unaffected.

2. Formulation

Consider a right circular cylinder of radius r_0 containing two layers of homogeneous immiscible fluid (see figure 1). The lighter layer with density ρ_1 lies above a denser layer with a density ρ_2 ($\rho_2 > \rho_1$). The thickness of each layer, H_1 and H_2, is assumed constant in the absence of motion relative to a co-ordinate system

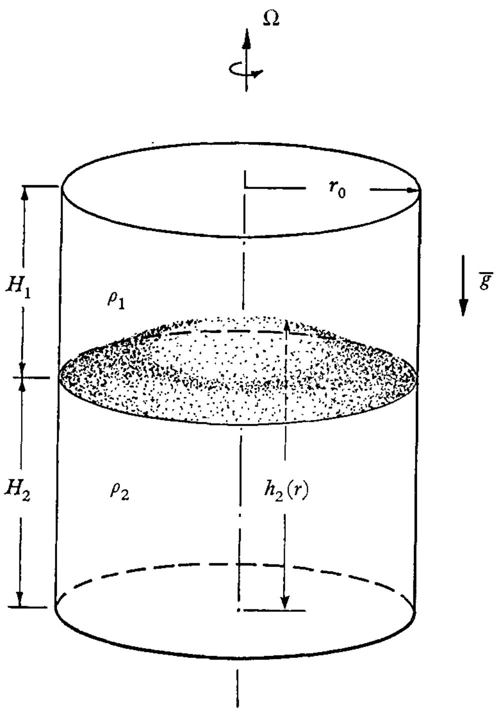


FIGURE 1. The configuration of the two-layer system.

rotating with an angular velocity Ω. This assertion, which ignores the centrifugal curvature of the interface between the fluids, will be valid whenever

$$\Omega^2 r_0^2/gH_n \ll 1 \quad (n = 1, 2),$$

a condition which will be assumed throughout this analysis.

The equations of motion describing the time-dependent motion of the fluid are

$$\frac{\partial \mathbf{q}_n}{\partial t} + \mathbf{q}_n . \nabla \mathbf{q}_n + 2\hat{k} \times \mathbf{q}_n = -\frac{\nabla p_n}{\rho_n} + \frac{\mu}{\rho_n} \nabla^2 \mathbf{q}_n - g\hat{k}, \tag{2.1}$$

$$\nabla . \mathbf{q}_n = 0. \tag{2.2}$$

The subscript $n = 1$ refers to the upper layer while $n = 2$ refers to the lower.

The boundary and initial conditions corresponding to an impulsive change $\epsilon\Omega$, $(\epsilon \ll 1)$, in the magnitude of the cylinder's angular velocity are $\mathbf{q}_n = \epsilon\Omega\hat{k}\times\mathbf{r}$ for $t \leqslant 0$ and $\mathbf{q}_n = 0$ on the solid boundaries. The unit vector $\hat{k}$ is parallel to the rotation axis. At the interface, $h_2(x, y, t)$, where continuity of velocity and stress is assumed, we have, in addition, the kinematical condition

$$\mathbf{q}_n . \hat{k} = dh_2/dt.$$

The following dimensionless variables, denoted by asterisks, are introduced:

$$\mathbf{r} = r_0\mathbf{r}_*, \quad t = \Omega^{-1}E^{-\frac{1}{2}}t_*, \quad \mathbf{q} = \epsilon r_0\Omega\mathbf{q}_*,$$

$$p_1 = \rho_1 g(H-z)+\rho_1\epsilon\Omega^2 r_0^2 p_{1*},$$

$$p_2 = \rho_1 gH+\rho_2 g(H_2-z)+\rho_2\epsilon\Omega^2 r_0^2 p_{2*},$$

$$h_2 = H_2[1+(\epsilon\Omega^2 r_0^2/g'H_2)\eta_2],$$

where

$$g' = (\rho_2-\rho_1)\,g/\rho_2 \quad \text{and} \quad g'/g \ll 1,$$

$$H = H_1+H_2 \quad \text{and} \quad E = \nu/\Omega r_0^2.$$

The density difference between the two layers is taken to be so slight that the kinematic viscosity is assumed to be the same in both layers of fluid.

If the upper surface, h, is a free surface, its position is given as

$$h = H[1+(\epsilon\Omega^2 r_0^2\eta/gH)]$$

while the kinematic condition (since $\Omega^2 r_0^2/gH \ll 1$) on $z = h$ is

$$\mathbf{q}_1 . \hat{k} = 0.$$

In terms of these variables the linearized equations of motion become (after dropping the asterisks)

$$E^{\frac{1}{2}}(\partial\mathbf{q}_n/\partial t)+2\hat{k}\times\mathbf{q}_n = -\nabla p_n+E\nabla^2\mathbf{q}_n \quad (n = 1,2), \tag{2.3}$$

$$\nabla . \mathbf{q}_n = 0, \tag{2.4}$$

with the linearized boundary and initial conditions

$$\mathbf{q}_n = 0 \quad \text{on rigid boundaries,} \tag{2.5}$$

$$\mathbf{q}_n . \hat{k} = E^{\frac{1}{2}}\frac{\Omega^2 r_0}{g'}\frac{\partial\eta_2}{\partial t} \quad \text{on} \quad z = H_2/r_0, \tag{2.6}$$

$$\mathbf{q}_n = \hat{k}\times\mathbf{r} \quad \text{for} \quad t \leqslant 0. \tag{2.7}$$

The conditions of continuity of stress and velocity at the interface will be discussed more fully later. Since the problem has axial symmetry all fields will be functions only of r, z and t. The velocity in each layer, $\mathbf{q}_n$, has components u_n, v_n, w_n in the radial, circumferential and vertical directions.

3. The technique of solution

The solution of the problem will be found in the case when $E \ll 1$, which obtains in most geophysically interesting cases, and which is easily reproduced in the laboratory. The solution in that case can be found by perturbation methods.

The solution in the interior of the fluid, removed from any boundary-layer region, can be written

$$\left.\begin{aligned} v_n &= v_n^{(0)}+E^{\frac{1}{2}}v_n^{(1)}+\ldots,\\ u_n &= E^{\frac{1}{2}}u_n^{(1)}+\ldots,\\ w_n &= E^{\frac{1}{2}}w_n^{(1)}+\ldots,\\ p_n &= p_n^{(0)}+E^{\frac{1}{2}}p_n^{(1)}+\ldots,\\ \eta_2 &= \eta_2^{(0)}+E^{\frac{1}{2}}\eta_2^{(1)}+\ldots.\end{aligned}\right\} \tag{3.1}$$

Substitution of (3.1) into (2.1) yields from the $O(1)$ equations:

$$2v_n^{(0)} = \partial p_n^{(0)}/\partial r, \tag{3.2}$$

$$0 = \partial p_n^{(0)}/\partial z, \tag{3.3}$$

which state that the $O(1)$ circumferential velocity is geostrophic and independent of z in the interior.

The $O(E^{\frac{1}{2}})$ problem yields the equations

$$\frac{\partial}{\partial t}v_n^{(0)} = -2u_n^{(1)}, \tag{3.4}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_n^{(1)})+\frac{\partial}{\partial z}w_n^{(1)} = 0, \tag{3.5}$$

or

$$\frac{\partial}{\partial t}\frac{1}{r}\frac{\partial}{\partial r}(rv_n^{(0)}) = 2\frac{\partial}{\partial z}w_n^{(1)}. \tag{3.6}$$

(3.6) states that the rate of change of the $O(1)$ vorticity is provided by vortex tube stretching parallel to the axis of rotation.

To complete the problem it is necessary to consider the Ekman layers which are needed to satisfy (2.5) on the rigid horizontal bounding surfaces. It can be shown (Greenspan & Howard 1963) that the effect of the layers (which have a non-dimensional thickness $O(E^{\frac{1}{2}})$) is to specify the vertical *interior* velocity at the rigid horizontal boundaries. These are

$$w_2^{(1)} = \frac{1}{2}\frac{\partial}{\partial r}(rv_2^{(0)}) \quad \text{on} \quad z = 0, \tag{3.7}$$

and

$$w_1^{(1)} = -\frac{1}{2}\frac{\partial}{\partial r}(rv_1^{(0)}) \quad \text{on} \quad z = \frac{H}{r_0}, \tag{3.8}$$

if the upper boundary is a rigid wall. If the upper surface is free, (3.8) is replaced by

$$w_1^{(1)} = 0 \quad \text{on} \quad z = H/r_0. \tag{3.9}$$

Similarly, Ekman layers at $z = H_2/r_0$ are required to satisfy the conditions of continuity of velocity and stress. The technique of matching is substantially the same as at a solid boundary and with (2.6) the conditions on the interior flow become

$$\left.\begin{aligned} w_1^{(1)} &= \frac{\Omega^2 r_0}{g'}\frac{\partial\eta_2^{(0)}}{\partial t}+\frac{1}{4r}\frac{\partial}{\partial r}(r[v_1^{(0)}-v_2^{(0)}]) \quad \text{on} \quad z = H_2/r_0,\\ w_2^{(1)} &= \frac{\Omega^2 r_0}{g'}\frac{\partial\eta_2^{(0)}}{\partial t}+\frac{1}{4r}\frac{\partial}{\partial r}(r[v_1^{(0)}-v_2^{(0)}]) \quad \text{on} \quad z = H_2/r_0.\end{aligned}\right\} \tag{3.10}$$

The $O(1)$ velocity is independent of z so that (3.6) may conveniently be vertically integrated in each layer to yield (using (3.2), (3.7), (3.8) and (3.10))

$$\frac{\partial}{\partial t}[D^2 p_1^{(0)} + k_1 F(p_2^{(0)} - p_1^{(0)})] = -k_1[D^2 p_1^{(0)} + \tfrac{1}{2}D^2(p_1^{(0)} - p_2^{(0)})], \tag{3.11a}$$

$$\frac{\partial}{\partial t}[D^2 p_2^{(0)} + k_2 F(p_1^{(0)} - p_2^{(0)})] = -k_2[D^2 p_2^{(0)} + \tfrac{1}{2}D^2(p_2^{(0)} - p_1^{(0)})], \tag{3.11b}$$

where

$$k_n = r_0/H_n \quad (n = 1, 2),$$
$$F = 4\Omega^2 r_0/g',$$
$$D^2 = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}.$$

In deriving (3.11 a, b) use has been made of the relation

$$\eta_2^{(0)} = p_2^{(0)} - p_1^{(0)},$$

which is derivable from (3.3) and the matching condition of the total pressure at $z = H_2/r_0$. If the upper surface is free at $z = H/r_0$ it is only necessary to change (3.11 a) by deleting the first term in the bracket on the right-hand side.

To complete the problem it is necessary to consider the side-wall boundary layers required to satisfy (2.5). For the layer problem considered here, the structure of the side-wall boundary layers is essentially the same as for the homogeneous problem. There is a layer existing in the region $(1-r) = O(E^{\frac{1}{4}})$ which brings the normal and circumferential velocities to rest, and a thinner layer in the region $(1-r) = O(E^{\frac{1}{3}})$ required to bring to rest the vertical velocity pumped out of the lower Ekman layer by the vertical $E^{\frac{1}{4}}$ layer (see Greenspan & Howard 1963).

The vertical velocity within the side-wall boundary layers is $O(E^{\frac{1}{4}})$. Thus, together with (2.6), this implies that the interface acts as a rigid lid preventing fluid from penetrating from one layer to the other within the side-wall boundary layer. This consideration is of paramount importance in understanding the relationship between the finite-layer models and the continuously stratified case. It is also possible to show by detailed consideration of the side-wall boundary layers that *no other condition is imposed by the boundary layers on the interior flow.* We shall see that this is true *only* for the problem involving the immiscible layers of fluid.

The correct condition on $p_n^{(0)}$ on $r = 1$ can be derived as follows. Since the motion field is axially symmetric it is necessary to explicitly pose the integral constraint that the *volume* of each layer of fluid is conserved. In terms of our non-dimensional variables this can be written

$$\frac{\partial}{\partial t}\int_0^1 r\eta_2^{(0)}\,dr = 0, \tag{3.12}$$

with an error of $O(E^{\frac{1}{4}})$ arising from the neglect of the volume of fluid in the side-wall boundary layers. In terms of the pressure field this condition is

$$\frac{\partial}{\partial t}\int_0^1 r(p_2^{(0)} - p_1^{(0)})\,dr = 0. \qquad (3.13$$

If (3.11) is integrated over the area of each fluid layer and (3.13) is used we find that

$$d\omega_1/dt = -k_1[\omega_1 + \tfrac{1}{2}(\omega_1 - \omega_2)], \tag{3.14a}$$

$$d\omega_2/dt = -k_2[\omega_2 + \tfrac{1}{2}(\omega_2 - \omega_1)], \tag{3.14b}$$

where
$$\omega_n(t) = \int_0^1 rD^2 p_n^{(0)} dr = 2v_n^{(0)}(1,t).$$

The application therefore of (3.13) to the equations of motion yields a pair of ordinary differential equations for the decay of the average vorticity in each layer, or, equivalently, for the decay of the circulation of the interior velocity at $r = 1$. *The spin up of the total vorticity in each layer is unaffected by baroclinic processes.* The baroclinic production of vorticity in each layer by vortex tube stretching must on the average be zero to conserve the volume of each layer. The baroclinicity may, however, yield important contributions to the vorticity change locally.

In addition (3.14) allows us to compute $v_n^{(0)}$ at $r = 1$ for all t and it provides a convenient recasting of the boundary condition (3.13) since $\partial p_n^{(0)}/\partial r$, on $r = 1$, is now determined by the solution of (3.14).†

4. Spin up for the free surface system

Consider the particular case where the upper surface at $z = H/r_0$ is free. The equations describing the decay of the $O(1)$ motion are then

$$(\partial/\partial t)[D^2 p_1^{(0)} + k_1 F(p_2^{(0)} - p_1^{(0)})] = -k_1[\tfrac{1}{2}D^2(p_1^{(0)} - p_2^{(0)})], \tag{4.1a}$$

$$(\partial/\partial t)[D^2 p_2^{(0)} + k_2 F(p_1^{(0)} - p_2^{(0)})] = -k_2[D^2 p_2^{(0)} + \tfrac{1}{2}D^2(p_2^{(0)} - p_1^{(0)})], \tag{4.1b}$$

while the equations governing the evolution of the average vorticity, or the interior rim velocity, are

$$dv_1^{(0)}(1,t)/dt = -\tfrac{1}{2}k_1[v_1^{(0)} - v_2^{(0)}], \tag{4.2a}$$

$$dv_2^{(0)}(1,t)/dt = -k_2^{(0)} + \tfrac{1}{2}(v_2^{(0)} - v_1^{(0)})], \tag{4.2b}$$

with the boundary and initial conditions

$$\partial p_n^{(0)}/\partial r = 2v_n^{(0)} = 2r \quad (t = 0); \tag{4.3a}$$

and
$$\partial p_n^{(0)}/\partial r = 2v_n^{(0)}(1,t) \quad (r = 1). \tag{4.3b}$$

The boundary condition on $p_n^{(0)}$ at $r = 1$ is given in terms of the solutions of (4.2*a*) and (4.2*b*). It is important to note before proceeding further the absolutely vital role of the interface frictional coupling. In the absence of this frictional coupling the right-hand side of (4.2*a*) would be zero, implying that in the absence of frictional coupling the average vorticity of the upper layer of this free surface system would not decay. In fact one would need to wait for viscous diffusion from the side walls to destroy the average vorticity, which effect takes a much longer time. (The effect of the side-wall diffusion is, in a spin-up time, still confined to a region of $O(E^{\frac{1}{4}})$ thick on the side wall.)

† Holton (1965) took $p_n^{(0)} = 0$ on $r = 1$ as his boundary condition.

The solutions of (4.2) which satisfy the conditions

$$v_n^{(0)}(1) = 1 \quad \text{at} \quad t = 0$$

are

$$v_1^{(0)}(1,t) = \frac{1}{(\alpha_2-\alpha_1)}(\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}), \tag{4.4a}$$

$$v_2^{(0)}(1,t) = \frac{1}{(\alpha_2-\alpha_1)}\left(\alpha_2\left(1+\frac{2\alpha_1}{k_1}\right)e^{\alpha_1 t} - \alpha_1\left(1+\frac{2\alpha_2}{k_1}\right)e^{\alpha_2 t}\right), \tag{4.4b}$$

where

$$\alpha_1 = -\tfrac{1}{4}(k_1+3k_2) + [\tfrac{1}{16}(k_1+3k_2)^2 - \tfrac{1}{2}k_1k_2]^{\frac{1}{2}},$$

$$\alpha_2 = -\tfrac{1}{4}(k_1+3k_2) - [\tfrac{1}{16}(k_1+3k_2)^2 - \tfrac{1}{2}k_1k_2]^{\frac{1}{2}}.$$

These results can then be used to construct the solutions to (4.1a) and (4.1b). The solution which satisfies (4.3a) and (4.3b) can be easily found, now that $\partial p_n^{(0)}/\partial r$ is known on $r = 1$, through the use of Laplace transforms and a Fourier–Bessel expansion and is

$$\begin{aligned} p_1^{(0)}(r,t) = {} & \frac{1}{(\alpha_2-\alpha_1)}[\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}](r^2 - \tfrac{1}{2}) \\ & - 8\alpha_1\alpha_2 F \sum_{m=1} \frac{J_0(\gamma_m r)}{\gamma_m^4 J_0(\gamma_m)[1+(k_1+k_2)(F/\gamma_m^2)]}\left[\frac{\alpha_1(\alpha_1+k_1)e^{+\alpha_1 t}}{(\alpha_1-\alpha_2)(\alpha_1-s_{1m})(\alpha_1-s_{2m})}\right. \\ & + \frac{\alpha_2(\alpha_2+k_2)e^{+\alpha_2 t}}{(\alpha_2-\alpha_1)(\alpha_2-s_{1m})(\alpha_2-s_{2m})} + \frac{s_{1m}(s_{1m}+k_2)e^{+s_{1m}t}}{(s_{1m}-\alpha_1)(s_{1m}-\alpha_2)(s_{1m}-s_{2m})} \\ & \left. + \frac{s_{2m}(s_{2m}+k_1)e^{+s_{2m}t}}{(s_{2m}-\alpha_1)(s_{2m}-\alpha_2)(s_{2m}-s_{1m})}\right], \end{aligned} \tag{4.5a}$$

$$\begin{aligned} p_2^{(0)}(r,t) = {} & \frac{1}{\alpha_2-\alpha_1}\left[\alpha_2\left(1+\frac{2\alpha_1}{k_1}\right)e^{\alpha_1 t} - \alpha_1\left(1+\frac{2\alpha_2}{k_1}\right)e^{\alpha_2 t}\right](r^2-\tfrac{1}{2}) \\ & + 8\alpha_1\alpha_2 F\frac{k_2}{k_1}\sum_{m=1}^{\infty}\frac{J_0(\gamma_m r)}{\gamma_m^4 J_0(\gamma_m)[1+(k_1+k_2)F/\gamma_m^2]}\left[\frac{\alpha_1 e^{+\alpha_1 t}}{(\alpha_1-\alpha_2)(\alpha_1-s_{1m})(\alpha_1-s_{2m})}\right. \\ & + \frac{\alpha_2^2 e^{+\alpha_2 t}}{(\alpha_1-\alpha_2)(\alpha_2-s_{1m})(\alpha_2-s_{2m})} + \frac{s_{1m}^2 e^{+s_{1m}t}}{(s_{1m}-\alpha_1)(s_{1m}-\alpha_2)(s_{1m}-s_{2m})} \\ & \left. + \frac{s_{2m}^2 e^{+s_{2m}t}}{(s_{2m}-\alpha_1)(s_{2m}-\alpha_2)(s_{2m}-s_{1m})}\right], \end{aligned} \tag{4.5b}$$

where

$$\begin{aligned} s_{1m} = {} & -\frac{[\tfrac{1}{2}(k_1+3k_2)+k_1k_2F/\gamma_m^2]}{2[1+(k_1+k_2)F/\gamma_m^2]} \\ & + \frac{[\tfrac{1}{2}(k_1+3k_2)+k_1k_2F/\gamma_n^2]^2 - 2k_1k_2[1+(k_1+k_2)F/\gamma_n^2]^{\frac{1}{2}}}{2[1+(k_1+k_2)F/\gamma_m^2]}, \end{aligned}$$

and

$$\begin{aligned} s_{2m} = {} & -\frac{[\tfrac{1}{2}(k_1+3k_2)+k_1k_2F/\gamma_m^2]}{2[1+(k_1+k_2)F/\gamma_m^2]} \\ & - \frac{\{[\tfrac{1}{2}(k_1+3k_2)+k_1k_2F/\gamma_m^2]^2 - 2k_1k_2[1+(k_1+k_2)F/\gamma_m^2]\}^{\frac{1}{2}}}{2[1+(k_1+k_2)F/\gamma_m^2]}, \end{aligned}$$

where γ_m is the mth zero of $J_1(\gamma_m)$.

5. Discussion of the layer system results

To simplify the discussion of the analytical results it is convenient to consider the case when both layers are of equal thickness, i.e. $k_1 = k_2 = k = 2r_0/H$. The qualitative nature of the result is unchanged but the resulting algebraic simplifications are considerable. In this case the decay rates α_1, α_2 for the total vorticity are simply

$$\alpha_1 = -k(1-1/\sqrt{2}) = -0{\cdot}3k,$$
$$\alpha_2 = -k(1+1/\sqrt{2}) = -1{\cdot}7k,$$

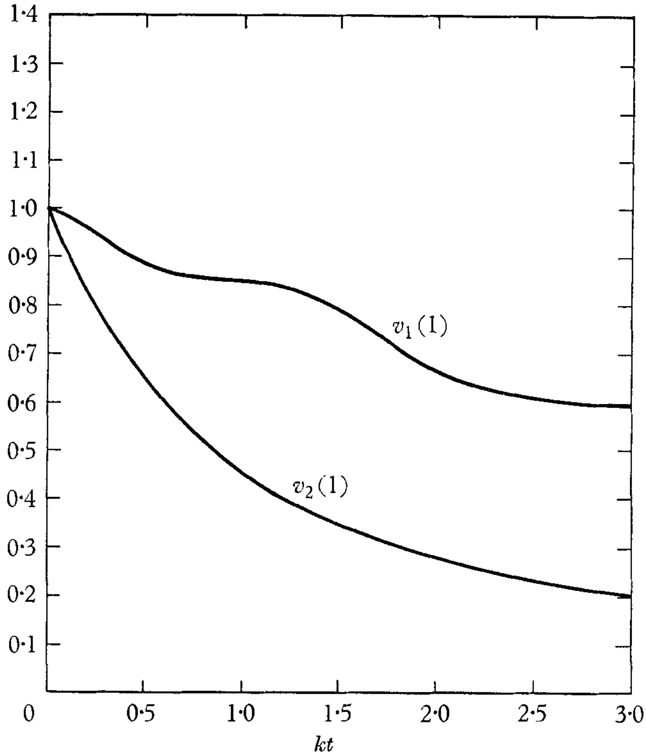


FIGURE 2. The time history of the circumferential velocities in each layer at $r = 1$.

while the rim velocities $v_n^{(0)}(1, t)$ are

$$\left.\begin{aligned} v_1^{(0)}(1,t) &+ 1{\cdot}2\,e^{-0{\cdot}3kt} - 0{\cdot}2\,e^{-1{\cdot}7kt}, \\ v_2^{(0)}(1,t) &= 0{\cdot}5\,e^{-0{\cdot}3kt} + 0{\cdot}5\,e^{-1{\cdot}7kt}, \end{aligned}\right\} \tag{5.1}$$

and are shown in figure 2. The vorticity in the lower layer which is directly acted upon by the frictional coupling to the lower surface spins out faster, leaving the upper fluid rotating until the velocity differences across the interface become large enough to frictionally couple the two layers and spin out the upper layer.

In the absence of baroclinic effects ($F = 0$) each layer would spin up as a solid body with an angular velocity in each layer given by $v_n^{(0)}(1,t)$. The pressure field would then be given by the first term in (4.5a) and (4.5b), i.e. excluding the

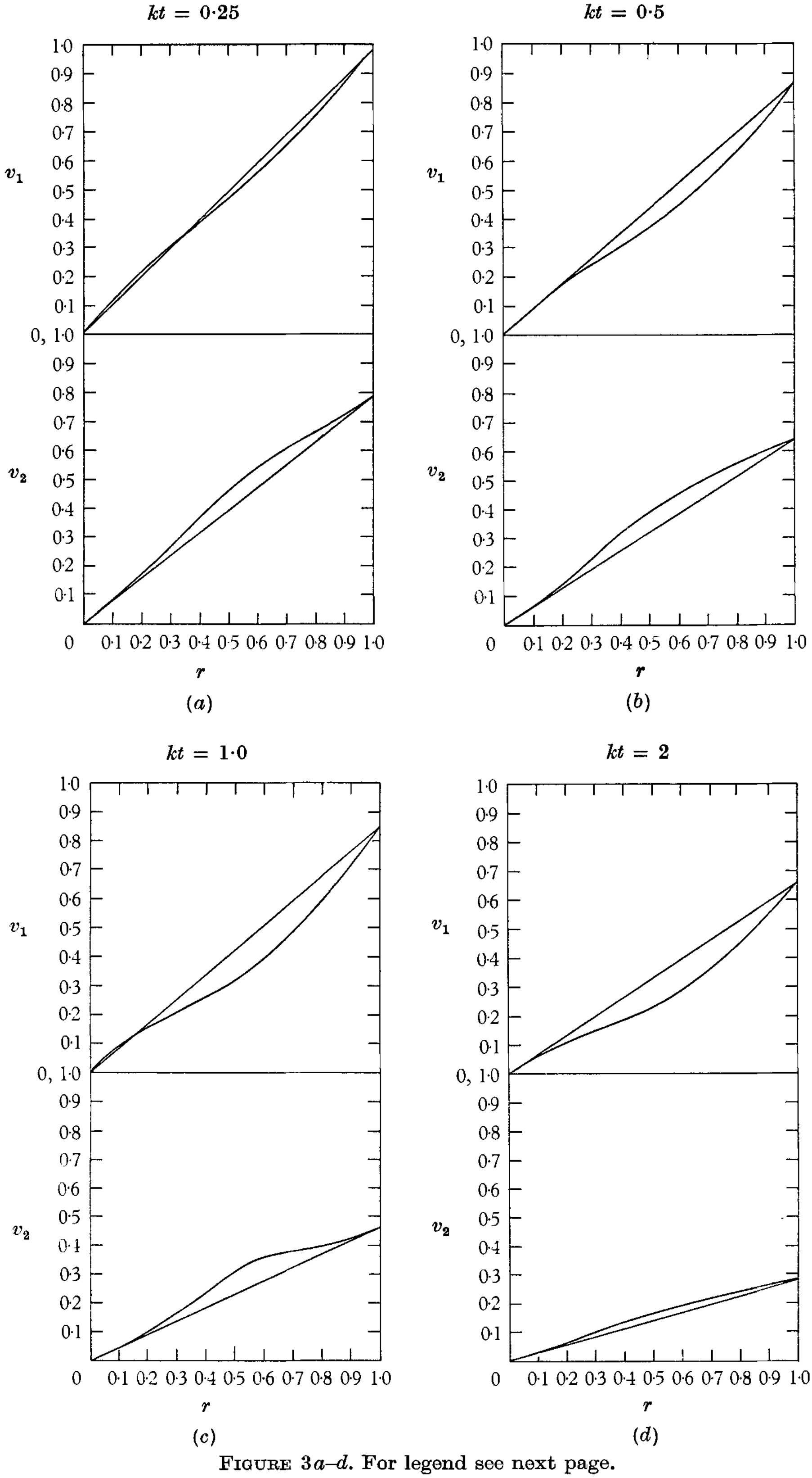
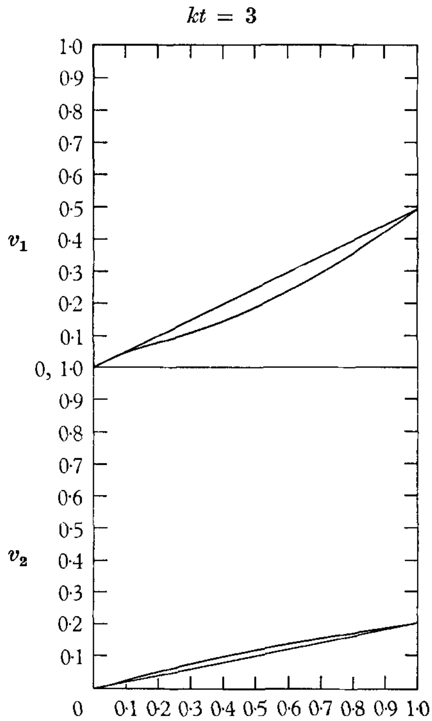


FIGURE 3*a*–*d*. For legend see next page.

Fourier–Bessel sum. The effect of the baroclinicity, and the *local* departure from this solid body spin, is given by the Fourier–Bessel series in (4.5*a*, *b*). For all non-zero, finite values of F the fluid in each layer will not spin up (or down) as a solid body. The radial structure of the velocity field at five characteristic times is



(*e*)

FIGURE 3*a*–*e*. The structure of the circumferential velocities in each layer. The straight lines show the circumferential velocities of a solid body rotation appropriate to the mean vorticity in each layer. The curved lines, which depict the true velocity structure, display the deviation from solid body rotation.

shown in figure 3 for a value of $kF = 15$. The departure from solid body rotation is slight for small kt but increases and becomes significant as time goes on. In fact, since it can easily be shown that

$$|s_{1m}| < |\alpha_1| < |s_{2m}| < |\alpha_2|,$$

the *slowest* root is s_{11} and for very large times the velocity field will tend toward the shape of the Bessel function $J_1(\gamma_1 r)$. However, for most reasonable values of F the velocity field is very small before this shape completely emerges. It is of interest to investigate the limit as $\rho_2 \to \rho_1$, i.e. when the fluid becomes homogeneous.

Noting that

$$\sum_{n=1}^{\infty} 4\frac{J_0(\gamma_n r)}{\gamma_n^2 J_0(\gamma_n)} = (r^2 - \tfrac{1}{2}),$$

it is easy to show in the limit $F \to \infty$ that

$$v_1^{(0)}(r,t) = v_2^{(0)}(r,t) = e^{-\frac{1}{2}kt}r,$$

which agrees with the result for a homogeneous fluid with a free surface.

The results presented here can be easily extended to a model with an arbitrarily large (but finite) number of immiscible fluid layers. It is important to note in such cases that, if the interfacial frictional coupling is absent, the total vorticity in any layer not adjacent to a horizontal rigid boundary would not reach the new state of solid body rotation in a spin-up time. If the upper layer is bounded above by a free surface, only the lowest layer of a multi-layer system would reach solid body rotation within a spin-up time scale $r_0(\nu\Omega)^{-\frac{1}{2}}$. This result is of prime importance in a discussion of the spin-up dynamics of a contained, continuously stratified fluid.

If the upper bounding surface for the layer model is rigid then the $O(1)$ motion is described by (3.11*a*, *b*) rather than (4.1*a*, *b*). In this case local baroclinic effects appear only when the layers are of different thicknesses. Otherwise the fluid spins up as if it were a homogeneous fluid. If the layers are of different depths the spin up will initially proceed at different rates in each layer, producing vertical shears across the fluid interfaces and introducing local baroclinic effects. Nevertheless, the average vorticity in each layer spins up independently of the local baroclinic effects and the technique of solution and the qualitative results are the same as those presented for the free surface model.

6. The spin up of a continuously stratified fluid

One might expect that the results of the analysis of the layer models would be applicable to the study of the continuously stratified model, especially if the number of layers considered is large. A little reflexion indicates the difficulties attendant on such an interpretation. The primary difficulty is that the continuous model restricts the role of viscosity to boundary-layer regions adjacent only to solid surfaces. The interfacial friction, which was seen to be vital in the layer models, is absent unless the effect of viscosity permeates the entire fluid, which is not plausible if $E \ll 1$. The implication would then be that a thermally non-conducting fluid (infinite Prandtl number) would not have its average angular velocity readjusted to the container's new speed until the much longer diffusion time ($t = O(E^{-1})$) obtained. Alternatively, this further suggests that the effect of thermal diffusion may be crucial. This can be understood physically in the following way. If the Prandtl number is small it will be very difficult for the fluid in the side-wall boundary layers to cross the surfaces of constant ambient density and hence it would be difficult to close the secondary flow required for spin up, *unless there is sufficient diffusion of density to break this constraint.* We noted in the layer model that the interface between two immiscible fluids acted as a rigid boundary preventing fluid mixing. In the continuous model the only way this difficulty can be overcome, as will be seen, is if the diffusion of density is important in the side-wall boundary layers. This will require a new side-wall boundary layer which is not present in the homogeneous or layer models.

The equations of motion in the case of the continuously stratified incompressible fluid are

$$(\partial \mathbf{q}/\partial t) + \mathbf{q}.\nabla\mathbf{q} + 2\hat{k}\times\mathbf{q} = -(\nabla p/\rho) + \nu\nabla^2\mathbf{q} - g\hat{k},$$
$$\nabla.\mathbf{q} = 0,$$
$$d\rho/dt = \kappa\nabla^2\rho,$$

which are substantially the same as in the layer case with the exception of the last equation relating to the Lagrangian density change to the rate of density (or temperature) diffusion. The diffusion coefficient is κ. The velocity, position, and time variables are non-dimensionalized as in § 2 and the density and pressure are written

$$\rho = \rho_0[1 + \sigma(z) + \epsilon F \rho_*],$$
$$p = -\rho_0 g z + \epsilon\Omega^2 r_0^2 \rho_0 p_*,$$

where $\sigma(z)$ is the variable part of the ambient density field. The fluid will be assumed to be Boussinesq and within this approximation the linearized equations describing the time-dependent motion of the fluid are

$$E^{\frac{1}{2}}(\partial\mathbf{q}/\partial t) + 2\hat{k}\times\mathbf{q} = -\nabla\mathbf{p} + E\nabla^2\mathbf{q} - \hat{k}\rho, \tag{6.1}$$
$$\nabla.\mathbf{q} = 0, \tag{6.2}$$
$$E^{\frac{1}{2}}(\partial\rho/\partial t) - wS^{-1} = \delta E\nabla^2\rho, \tag{6.3}$$

where $S = -\Omega^2 r_0/(\partial\sigma/\partial z)g = O(1)$, and $\delta = \kappa/\nu$.

Note that S (which will be considered a constant throughout) is positive for the stably stratified fluids considered here. The parameter δ will be considered an $O(1)$ parameter with respect to E.

As before the interior velocity, pressure (and now also density) will be found, for $E \ll 1$, by a perturbation expansion

$$\left.\begin{aligned} v &= v^{(0)} + E^{\frac{1}{2}}v^{(1)} + \dots,\\ u &= E^{\frac{1}{2}}u^{(1)} + \dots,\\ w &= E^{\frac{1}{2}}w^{(1)} + \dots,\\ p &= p^{(0)} + E^{\frac{1}{2}}p_1 + \dots,\\ \rho &= \rho^{(0)} + E^{\frac{1}{2}}\rho_1 + \dots. \end{aligned}\right\} \tag{6.4}$$

Substitution of (6.4) into (6.1), (6.2) and (6.3) yields the zero-order relations

$$\left.\begin{aligned} \partial p^{(0)}/\partial r &= v^{(0)},\\ \partial p^{(0)}/\partial z &= -\rho^{(0)}. \end{aligned}\right\} \tag{6.5}$$

The $O(E^{\frac{1}{2}})$ equations, with a little manipulation, can be shown to yield

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv^{(0)}) - 2S\frac{\partial}{\partial z}\rho^{(0)}\right) = 0. \tag{6.6}$$

This latter equation is the continuous analogue of (3.6). With the use of (6.5), (6.6) may be written completely in terms of $p^{(0)}$, i.e.

$$\frac{\partial}{\partial t}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p^{(0)}}{\partial r}\right) + 4S\frac{\partial^2}{\partial z^2}p^{(0)}\right] = 0. \tag{6.7}$$

The boundary conditions on $z = 0$, H/r_0 are

$$\mathbf{q} = 0 \tag{6.8}$$

and some similar condition on the density; either

$$\rho = 0 \tag{6.9}$$

if the surfaces are held at constant temperature (density) or

$$\hat{k}.\nabla\rho = 0 \tag{6.10}$$

if the surfaces are insulated.

In either case it can be shown that the condition of dynamical consequence is (6.8). This condition is again satisfied by the application of Ekman boundary layers whose structure is, to this order, unchanged by the presence of stratification. The condition that the boundary-layer analysis places on the interior flow is again

$$w^{(1)} = \frac{1}{2}\frac{\partial}{\partial r}(rv^{(0)}) \quad \text{on} \quad z = 0, \tag{6.11}$$

and

$$w^{(1)} = -\frac{1}{2}\frac{\partial}{\partial r}(rv^{(0)}) \quad \text{on} \quad z = H/r_0. \tag{6.12}$$

With the application of (6.5) and (6.3) these conditions can be written in terms of the pressure as

$$4S\frac{\partial^2 p^{(0)}}{\partial z\,\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial p^{(0)}}{\partial r} \quad \text{on} \quad z = 0, \tag{6.13a}$$

$$4S\frac{\partial^2 p^{(0)}}{\partial z\,\partial t} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial p^{(0)}}{\partial r} \quad \text{on} \quad z = H/r_0. \tag{6.13b}$$

These equations provide the necessary boundary conditions for (6.7) on the horizontal boundary surfaces. The conditions (6.9) or (6.10) can be satisfied through the application of a boundary layer of thickness $E^{\frac{1}{4}}$. This boundary layer is essentially a layer produced by the diffusion of temperature or density into the interior within a spin up time. It does not dynamically affect the interior motion and will not be discussed further.

To proceed, some condition on $p^{(0)}$ must be set on $r = 1$. First note the following. If the Prandtl number is infinite ($\delta = 0$) and diffusion of density neglected from the beginning, the condition of mass conservation (or an integral statement of (5.2)) would imply that for any $0 < z < H/r_0$,

$$\int_0^1 rw\,dr = 0$$

and therefore, through the application of (6.3), that

$$\frac{\partial}{\partial t}\int_0^1 \rho r\,dr = 0. \tag{6.14}$$

If the density perturbation in any side-wall layer is assumed less than $O(E^{-\frac{1}{2}})$, then the density in (6.14) can be replaced by $\rho^{(0)}$. (The restriction that ρ is less than $O(E^{-\frac{1}{2}})$ is certainly plausible; the reason for the restriction will become clear below.) In that event (6.14) applied to (6.6) yields

$$\partial r v^{(0)}/\partial t = 0 \quad \text{on} \quad r = 1, \tag{6.15}$$

which states that the *average* vorticity at any z (or the circulation on a contour $r = 1$) is not dissipated within a spin up time. This result is surprising but consistent with the physical reasoning given in the beginning of this section. To remove this difficulty it is necessary to consider the side wall boundary layers in detail with the *retention of the density diffusion* ($\delta \neq 0$).

It is found that the boundary-layer structure that emerges on the side walls has a different character than that present in the homogeneous case. An $E^{\frac{1}{4}}$ layer still exists but in an altered form. Let the dependent variables in this layer be expressed as

$$\left.\begin{aligned} v &= v^{(0)}(r,z,t) + E^{\frac{1}{2}} v^{(1)} + \ldots + V(\eta,z,t) + \ldots, \\ u &= E^{\frac{1}{2}} u^{(0)}(r,z,t) + \ldots + EU(\eta,z,t) + \ldots, \\ w &= E^{\frac{1}{2}} w^{(0)}(r,z,t) + \ldots, \\ \rho &= \rho^{(0)}(r,z,t) + E^{\frac{1}{2}} \rho^{(1)} + \ldots + E^{\frac{1}{4}} \theta(\eta,z,t) + \ldots, \\ p &= p^{(0)}(r,z,t) + E^{\frac{1}{2}} p^{(1)}(r,z,t) + E^{\frac{1}{4}} P(\eta,z,t), \end{aligned}\right\} \tag{6.16}$$

where $\eta = (1-r)E^{-\frac{1}{4}}$.

If (6.16) is substituted into (6.1), (6.2) and (6.3) and the limit $E \to 0$ for fixed η is taken, the resulting equations yield

$$V_t = V_{\eta\eta}, \tag{6.17}$$

which is a diffusion equation for V. The solution can be used to satisfy the no slip condition on the circumferential velocity at $r = 1$. It places no condition on the interior flow and will not be considered further.

It is still necessary to satisfy the no slip condition on u and w. In addition conditions on the density must be specified on $r = 1$. If the side wall is held at a fixed temperature,

$$\rho = 0 \quad \text{on} \quad r = 1; \tag{6.18}$$

while if the side walls are insulated

$$\partial \rho/\partial r = 0 \quad \text{on} \quad r = 1. \tag{6.19}$$

In the homogeneous and layer problems, the remaining velocity fields are brought to rest within an inner layer whose thickness is $O(E^{\frac{1}{3}})$. No such layer exists for the continuously stratified system governed by (6.1), (6.2) and (6.3).

If (6.18) is the condition given on the density at $r = 1$, then the remaining boundary conditions are satisfied within a layer of order $E^{\frac{1}{2}}$.†

Let the dependent variables within this layer be expressed as

$$\left.\begin{aligned} v &= v^{(0)}(r,z,t) + V(\eta,z,t) + E^{\frac{1}{2}} V_s(x,z,t) + \ldots, \\ u &= E^{\frac{1}{2}}(u^{(0)}(r,z,t) + U_s(x,z,t) + \ldots, \\ w &= W_s(x,z,t) + \ldots, \\ \rho &= \theta_s(x,z,t) + \ldots + \rho^{(0)}(r,z,t) + \ldots, \\ p &= p^{(0)}(r,z,t) + E^{\frac{1}{4}} P(\eta,z,t) + \ldots, \end{aligned}\right\} \tag{6.20}$$

where $x = (1-r)E^{-\frac{1}{2}}$.

† It is for this reason that the condition $\rho < O(E^{-\frac{1}{2}})$ was required to derive (6.15).

The dots in (6.20) refer to terms of higher order in E which are not important for this discussion. All the s subscripted variables are chosen such that

$$\lim_{x\to\infty} (\;)_s = 0.$$

If (6.20) is substituted into (6.1), (6.2) and (6.3) the following equations obtain in the limit $E \to 0$, for fixed x,

$$2U_s = \partial^2 V_s/\partial x^2, \tag{6.21}$$

$$\partial U_s/\partial x = \partial W_s/\partial z, \tag{6.22}$$

$$\theta_s = \partial^2 W_s/\partial x^2, \tag{6.23}$$

$$-W_s = \delta S(\partial^2\theta_s/\partial x^2). \tag{6.24}$$

It is interesting to note that the variables θ_s and W_s play the same role in this thermal side-wall boundary layer that the two components of horizontal velocity play in the Ekman layers which exist on the horizontal bounding surfaces. The relevant solutions, which possess the characteristic Ekman spiral behaviour, are

$$\theta_s = e^{-x/\beta}\{A(z,t)\cos(x/\beta) + B(z,t)\sin(x/\beta)\}, \tag{6.25}$$

$$W_s = -\tfrac{1}{2}\beta^2 e^{-x/\beta}\{A\sin(x/\beta) - B\cos(x/\beta)\}, \tag{6.26}$$

$$U_s = \frac{\beta^3}{4}e^{-x/\beta}\left\{\frac{\partial A}{\partial z}\left[\sin\frac{x}{\beta} + \cos\frac{x}{\beta}\right] + \frac{\partial B}{\partial z}\left[\sin\frac{x}{\beta} - \cos\frac{x}{\beta}\right]\right\}, \tag{6.27}$$

where
$$\beta = \sqrt{2}(\delta S)^{\frac{1}{4}}.$$

The boundary conditions to be satisfied on $x = 0$ are

$$W_s = 0, \tag{6.28a}$$

$$\theta_s + p^{(0)} = 0, \tag{6.28b}$$

$$U_s + u^{(0)} = 0. \tag{6.28c}$$

The application of (6.28a) yields

$$B(z) = 0,$$

while (6.28b) and (6.28c) along with (6.5) and the interior $O(E^{\frac{1}{2}})$ equations which lead to (6.6), in turn yield the conditions

$$A(z) = \partial p^{(0)}/\partial z,$$

$$\beta^3(\partial A/\partial z) = \partial^2 p^{(0)}/\partial t\,\partial r,$$

which combine to yield as a condition on the *interior flow* at $r = 1$

$$\partial^2 p^{(0)}/\partial t\,\partial r = \beta^3(\partial^2 p^{(0)}/\partial z^2). \tag{6.29}$$

It is interesting to note the similarity in structure of the side wall condition (6.29) with the conditions (6.13a, b) set by the Ekman layers at $z = 0, H/r_0$. If the thermal diffusion is ignored ($\delta = 0$) then $\beta = 0$, and (6.29) implies that

$$\partial v^{(0)}/\partial t = 0 \quad \text{on} \quad r = 1,$$

i.e. that the total vorticity at each level (or the circulation at $r = 1$) is time independent on a spin up time scale. Only the retention of the thermal diffusion process allows the mean vorticity to adjust to the new container's angular velocity. The interior problem which consists of (6.7), the boundary conditions (6.13*a*, *b*) and (6.29), and the initial condition

$$\frac{\partial p^{(0)}}{\partial r} = 2r \quad \text{at} \quad t = 0,$$

is in general formidable. Nevertheless, a particularly simple solution can be found in the special case when

$$\beta^3 = 4Sr_0/H.$$

In that case the solution is

$$p^{(0)} = \{r^2 - [z - (H/2r_0)]^2/2S\}e^{-2r_0t/H}. \tag{6.30}$$

The initial state spins out as a solid body in the time required for a homogeneous fluid to do the same, with however an alteration of the density field which increases the static stability of the fluid slightly. This is undoubtedly a very special solution which illustrates dramatically the essential differences between the layered and continuously stratified fluid problems.

The essential difference between the continuously stratified and the layered, or homogeneous fluids is that the former depends strongly on the ability of the fluid to diffuse density sufficiently rapidly to allow the vertical flux of fluid in the side-wall layers to close the secondary meridional flow. When this is not possible, as in the infinite Prandtl number case ($\delta = 0$), the fluid must wait for viscous diffusion from the side walls to slowly work its way into the interior. More striking is the example of insulated side walls, where the boundary condition (6.19) applies. The same $O(E^{\frac{1}{2}})$ boundary layer obtains, but the inability of the fluid to diffuse temperature (density) through the side walls is now reflected in the fact that θ_s and $W_s = O(E^{\frac{1}{2}})$ so that the secondary circulation is not closed in this layer. The application of (6.28*c*), then further implies (since $U_s = O(E)$) that

$$\frac{\partial}{\partial t}\frac{\partial p^{(0)}}{\partial r} = 0 \quad \text{on} \quad r = 1,$$

so that for the insulated fluid ($\rho_r = 0$ on $r = 1$) the inviscid circulation for all $O(1)$ Prandtl numbers is preserved in a spin up time scale, and viscous side-wall diffusion must be invoked. This suggests a rescaling of the time. Let the time be scaled

$$t_{\text{dimensional}} = \Omega^{-1}E^{-1}t.$$

This scaling assumes that dissipative processes, rather than Ekman layer suction are important in the fluid interior. The resulting dynamical equations become

$$E(\partial \mathbf{q}/\partial t) + 2\hat{k}\times\mathbf{q} = -\nabla p + E\nabla^2\mathbf{q} - \rho\hat{k}, \tag{6.31}$$

$$\nabla . \mathbf{q} = 0, \tag{6.32}$$

$$E(\partial\rho/\partial t) - wS^{-1} = \delta E\nabla^2\rho. \tag{6.33}$$

If $\mathbf{q}$ and p are expanded in powers of E we obtain

$$2v^{(0)} = -\partial p^{(0)}/\partial r, \tag{6.34}$$

$$\rho^{(0)} = -\partial p^{(0)}/\partial z, \tag{6.35}$$

so that the $O(1)$ variables are still hydrostatic and geostrophic. However, a consideration of the $O(E)$ equations yields an equation for $p^{(0)}$,

$$\frac{\partial}{\partial t}\left\{D^2 p^{(0)} + 4S\frac{\partial^2 p^{(0)}}{\partial z^2}\right\} = \left\{D^2 + \frac{\partial^2}{\partial z^2}\right\}\left\{D^2 p^{(0)} + \frac{4\kappa}{\nu} S \frac{\partial^2 p^{(0)}}{\partial z^2}\right\}, \quad (6.36)$$

which states that the rate of change of the interior potential vorticity is due to the effects of *diffusion in the interior* by viscosity and *heat conduction*. The solution for the insulated spin up problem which satisfies (6.36) and the conditions:

$$\begin{aligned} \partial p^{(0)}/\partial r &= 0 \quad \text{on} \quad r = 1; \\ \partial^2 p^{(0)}/\partial z\,\partial r &= 0 \quad \text{on} \quad r = 1; \\ \partial p^{(0)}/\partial r &= 0 \quad \text{on} \quad z = 0, 1; \\ \partial^2 p^{(0)}/\partial z^2 &= 0 \quad \text{on} \quad z = 0, 1; \\ \partial p^{(0)}/\partial r &= 2r \quad \text{for} \quad t = 0; \end{aligned}$$

can be shown to be

$$p^{(0)} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\left\{\exp - \left[\frac{(n^2\pi^2 + \gamma^2 m)(\nu n^2\pi^2 + \gamma_m^2 \kappa 4S)/\nu}{(n^2\pi^2 + 4S\gamma_n^2)}\right] t\right\}\sin \pi z J_0(\gamma_m r), \quad (6.37)$$

where $$A_{mn} = \frac{8[1-(-1)^n]}{n\pi\gamma_n^2 J_0(\gamma_n)}$$

and $$J_0(\gamma_n) = 0.$$

Thus, according to the linear theory in the insulated problem there are no $O(1)$ Ekman layers and no side-wall boundary layers; the interior 'spins up' by a strictly diffusive process in a time scale $O(E^{-1})$. The continuously stratified fluid, when insulated, then behaves in a dramatically different manner than the homogeneous fluid. The secondary circulations are so inhibited by the circulation that no Ekman layer suction is possible.

Similar problems involving continuous stratification are being studied further.

This research was partially supported by the Office of Scientific Research of the U.S. Air Force, Grant AF-AFOSR-492-66.

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